

Electromagnetic scattering by a fixed finite object embedded in an absorbing medium

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Abstract: This paper presents a general and systematic analysis of the problem of electromagnetic scattering by an arbitrary finite fixed object embedded in an absorbing, homogeneous, isotropic, and unbounded medium. The volume integral equation is used to derive generalized formulas of the far-field approximation. The latter serve to introduce direct optical observables such as the phase and extinction matrices. The differences between the generalized equations and their counterparts describing electromagnetic scattering by an object embedded in a non-absorbing medium are discussed.

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References and links

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1. Introduction

The past decade has revealed a growing interest in the problem of electromagnetic scattering by particles embedded in an absorbing host medium (see, e.g., [1–7] and references therein). The objective of the majority of publications on this subject has been to argue what modified quantities should serve in lieu of their conventional analogs appearing in the theories of single scattering and radiative transfer by spherical particles embedded in a non-absorbing host medium [8–14]. The intent of this paper and its multiple-scattering companion [15] is somewhat different. The major objective is to perform as general an analysis of electromagnetic scattering as possible without providing a detailed microphysical specification of the scattering object and entering it into a numerical solver of the Maxwell equations. I will assume, in particular, that particles are arbitrarily sized, shaped, and oriented. The main subjects of this paper are single scattering by a fixed finite object, the far-field approximation, and reciprocity relations. In [15], I will analyze the problem of multiple scattering and provide a direct microphysical derivation of the radiative transfer equation which yields unambiguous formulas for the participating quantities. The common philosophy adopted here as well as in [15] is that a theoretical quantity is worth being introduced only if it describes an actual optical observable or enters explicitly a formula for an actual optical observable.

To achieve this overall objective, I will parallel the systematic analysis of electromagnetic scattering presented in [13, 14] with emphasis on the specific changes caused by letting the host medium be absorbing. I will assume that the reader has access to [13, 14] and will use the same terminology and notation.

2. Volume integral equation

Consider a fixed scattering object embedded in an infinite, homogeneous, linear, isotropic, and potentially absorbing host medium. The scatterer occupies a finite interior region V_{INT} and is surrounded by the infinite exterior region V_{EXT} such that $V_{\text{INT}} \cup V_{\text{EXT}} = \mathcal{R}^3$, where \mathcal{R}^3 denotes the entire three-dimensional space. The interior region is filled with an isotropic, linear, and possibly inhomogeneous material. The scatterer can be either a single body or a cluster with touching and/or separated components. Point O serves as the common origin of all position vectors \mathbf{r} and as the origin of the laboratory coordinate system (Fig. 1).

The frequency-domain monochromatic Maxwell curl equations describing the scattering problem are as follows [16]:

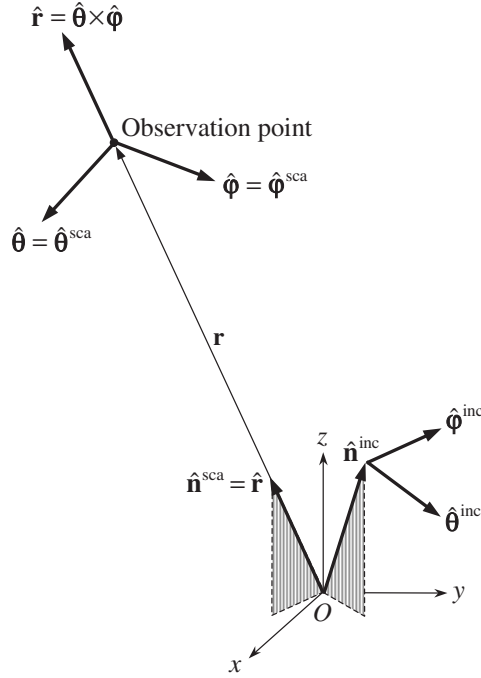


Fig. 1. Electromagnetic scattering by a fixed object.

$$\left. \begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= i\omega\mu_1 \mathbf{H}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) &= -i\omega\varepsilon_1 \mathbf{E}(\mathbf{r}) \end{aligned} \right\} \quad \mathbf{r} \in V_{\text{EXT}}, \quad (1)$$

$$\left. \begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= i\omega\mu_2 \mathbf{H}(\mathbf{r}) \\ \nabla \times \mathbf{H}(\mathbf{r}) &= -i\omega\varepsilon_2(\mathbf{r}) \mathbf{E}(\mathbf{r}) \end{aligned} \right\} \quad \mathbf{r} \in V_{\text{INT}}, \quad (2)$$

where $i = (-1)^{1/2}$, \mathbf{E} is the electric field, \mathbf{H} is the magnetic field, ω is the angular frequency, μ_1 and ε_1 are the permeability and complex permittivity of the host medium, and μ_2 and $\varepsilon_2(\mathbf{r})$ are the permeability and complex permittivity of the scattering object. Note that ε_2 is allowed to vary throughout the scattering object, whereas ε_1 , μ_1 , and μ_2 are assumed to be arbitrary but constant. Since the first relations in Eqs. (1) and (2) yield the magnetic field provided that the electric field is known everywhere, we will look for the solution of Eqs. (1) and (2) in terms of only the electric field. The latter satisfies the following vector wave equations:

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_1^2 \mathbf{E}(\mathbf{r}) = 0, \quad \mathbf{r} \in V_{\text{EXT}}, \quad (3)$$

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_2^2(\mathbf{r}) \mathbf{E}(\mathbf{r}) = 0, \quad \mathbf{r} \in V_{\text{INT}}, \quad (4)$$

where the wave numbers of the exterior and interior regions, $k_1 = \omega[\varepsilon_1\mu_1]^{1/2}$ and $k_2(\mathbf{r}) = \omega[\varepsilon_2(\mathbf{r})\mu_2]^{1/2}$, are, in general, complex-valued. It is convenient for our purposes to represent the wave number of the host medium in terms of its real and imaginary parts: $k_1 = k'_1 + ik''_1$, where $k'_1 > 0$ and $k''_1 \geq 0$. Equations (3) and (4) can be rewritten as a single inhomogeneous differential equation

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k_1^2 \mathbf{E}(\mathbf{r}) = \mathbf{j}(\mathbf{r}), \quad \mathbf{r} \in \mathfrak{R}^3, \quad (5)$$

where

$$\mathbf{j}(\mathbf{r}) = k_1^2 [\tilde{m}^2(\mathbf{r}) - 1] \mathbf{E}(\mathbf{r}), \quad (6)$$

$$\tilde{m}(\mathbf{r}) = \begin{cases} 1, & \mathbf{r} \in V_{\text{EXT}}, \\ m(\mathbf{r}) = k_2(\mathbf{r})/k_1, & \mathbf{r} \in V_{\text{INT}}, \end{cases} \quad (7)$$

and $m(\mathbf{r})$ is the refractive index of the interior relative to that of the exterior. The complex wave numbers and the relative refractive index are, in general, frequency-dependent. It follows from Eq. (6) that the forcing function $\mathbf{j}(\mathbf{r})$ vanishes everywhere outside the interior region.

Repeating step-by-step the derivation outlined in Section 2.1 of [13] or Section 3.1 of [14], one can show that the general physically appropriate solution of Eq. (5) can be written in terms of the solution of the conventional volume integral equation (VIE) [17]:

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{inc}}(\mathbf{r}) + k_1^2 \left(\tilde{I} + \frac{1}{k_1^2} \nabla \otimes \nabla \right) \cdot \int_{V_{\text{INT}}} d\mathbf{r}' [m^2(\mathbf{r}') - 1] \mathbf{E}(\mathbf{r}') \frac{\exp(ik_1|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|}, \quad \mathbf{r} \in \mathfrak{R}^3, \quad (8)$$

where $\mathbf{E}^{\text{inc}}(\mathbf{r})$ is the incident field, \tilde{I} is the identity dyadic, \otimes is the dyadic product sign, the dot is the scalar product sign, and the wave number of the host medium is now explicitly allowed to be complex-valued. An essential ingredient of this derivation is the fact that the physically appropriate solution of the Helmholtz equation $(\nabla^2 + k_1^2)g(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}')$ with a complex k_1 is $g(\mathbf{r}, \mathbf{r}') = \exp(ik_1|\mathbf{r} - \mathbf{r}'|)/4\pi|\mathbf{r} - \mathbf{r}'|$ [18], where $\delta(\mathbf{r} - \mathbf{r}')$ is the three-dimensional delta function. The second term on the right-hand side of Eq. (8) satisfies the requisite Sommerfeld radiation condition at infinity [18] and gives the scattered field $\mathbf{E}^{\text{sca}}(\mathbf{r})$. The latter can be expressed in terms of the incident field as follows:

$$\mathbf{E}^{\text{sca}}(\mathbf{r}) = \left(\tilde{I} + \frac{1}{k_1^2} \nabla \otimes \nabla \right) \cdot \int_{V_{\text{INT}}} d\mathbf{r}' \frac{\exp(ik_1|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|} \int_{V_{\text{INT}}} d\mathbf{r}'' \tilde{T}(\mathbf{r}', \mathbf{r}'') \cdot \mathbf{E}^{\text{inc}}(\mathbf{r}''), \quad \mathbf{r} \in \mathfrak{R}^3, \quad (9)$$

where \tilde{T} is the dyadic transition operator satisfying the following integral equation:

$$\begin{aligned} \tilde{T}(\mathbf{r}, \mathbf{r}') &= k_1^2 [m^2(\mathbf{r}) - 1] \delta(\mathbf{r} - \mathbf{r}') \tilde{I} \\ &+ k_1^2 [m^2(\mathbf{r}) - 1] \left(\tilde{I} + \frac{1}{k_1^2} \nabla \otimes \nabla \right) \cdot \int_{V_{\text{INT}}} d\mathbf{r}'' \frac{\exp(ik_1|\mathbf{r} - \mathbf{r}''|)}{4\pi|\mathbf{r} - \mathbf{r}''|} \tilde{T}(\mathbf{r}'', \mathbf{r}'), \quad \mathbf{r}, \mathbf{r}' \in V_{\text{INT}}. \end{aligned} \quad (10)$$

Again, this equation does not differ mathematically from Eq. (2.18) in [13] or Eq. (3.1.24) in [14], but now the absorptivity of the host medium can affect its solution via the modified relative refractive index, Eq. (7), and the explicitly complex-valued wave number of the host medium.

3. Scattering in the far-field zone

Let us now choose the origin O close to the geometrical center of the scattering object and assume that the distance r from O to the observation point \mathbf{r} (Fig. 1) is much greater than any linear dimension of the scattering object:

$$r \gg r' \text{ for any } \mathbf{r}' \in V_{\text{INT}}, \quad (11)$$

where $r = |\mathbf{r}|$ and $r' = |\mathbf{r}'|$. We also assume that

$$\frac{|k_1|r'^2}{2r} \ll 1 \text{ for any } \mathbf{r}' \in V_{\text{INT}}. \quad (12)$$

Since

$$|\mathbf{r} - \mathbf{r}'| = r \sqrt{1 - 2 \frac{\hat{\mathbf{r}} \cdot \mathbf{r}'}{r} + \frac{r'^2}{r^2}} \approx r - \hat{\mathbf{r}} \cdot \mathbf{r}' + \frac{r'^2}{2r}, \quad (13)$$

where $\hat{\mathbf{r}} = \mathbf{r}/r$ is the unit vector in the direction of \mathbf{r} (Fig. 1), we have

$$\frac{\exp(ik_1|\mathbf{r} - \mathbf{r}'|)}{4\pi|\mathbf{r} - \mathbf{r}'|} \approx \frac{\exp(ik_1r - ik_1\hat{\mathbf{r}} \cdot \mathbf{r}')}{4\pi r}. \quad (14)$$

Finally, assuming that

$$|k_1|r \gg 1, \quad (15)$$

we obtain from Eqs. (8) and (14)

$$\begin{aligned} \mathbf{E}^{\text{sca}}(\mathbf{r}) &\approx \frac{\exp(ik_1r)}{r} \mathbf{E}_1^{\text{sca}}(\hat{\mathbf{r}}) \\ &= \exp(-k_1''r) \frac{\exp(ik_1'r)}{r} \mathbf{E}_1^{\text{sca}}(\hat{\mathbf{r}}), \end{aligned} \quad (16)$$

where the distance-independent amplitude of the transverse outgoing spherical wave is given by

$$\mathbf{E}_1^{\text{sca}}(\hat{\mathbf{r}}) = \frac{k_1^2}{4\pi} (\vec{I} - \hat{\mathbf{r}} \otimes \hat{\mathbf{r}}) \cdot \int_{V_{\text{INT}}} d\mathbf{r}' [m^2(\mathbf{r}') - 1] \mathbf{E}(\mathbf{r}') \exp(-ik_1\hat{\mathbf{r}} \cdot \mathbf{r}'), \quad \hat{\mathbf{r}} \cdot \mathbf{E}_1^{\text{sca}}(\hat{\mathbf{r}}) = 0. \quad (17)$$

The generalized far-field equations (16) and (17) show that the scattered field is attenuated not only by the factor $1/r$ but also by the exponential absorption factor $\exp(-k_1''r)$.

Let us now assume that the incident field is a *homogeneous* plane electromagnetic wave given by

$$\mathbf{E}^{\text{inc}}(\mathbf{r}) = \exp(ik_1\hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{r}) \mathbf{E}_0^{\text{inc}}, \quad \mathbf{E}_0^{\text{inc}} \cdot \hat{\mathbf{n}}^{\text{inc}} = 0. \quad (18)$$

Note that $\mathbf{E}_0^{\text{inc}}$ is the electric field at the origin of the laboratory coordinate system. We then have

$$\mathbf{E}_1^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}}) = \vec{A}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \cdot \mathbf{E}_0^{\text{inc}}, \quad (19)$$

where $\hat{\mathbf{n}}^{\text{sca}} = \hat{\mathbf{r}}$ is the scattering direction (see Fig. 1), \vec{A} is the scattering dyadic such that

$$\hat{\mathbf{n}}^{\text{sca}} \cdot \vec{A}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) = \mathbf{0} \quad \text{and} \quad \vec{A}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \cdot \hat{\mathbf{n}}^{\text{inc}} = \mathbf{0}, \quad (20)$$

and $\mathbf{0}$ is a zero vector. The expression for the scattering dyadic in terms of the dyadic transition operator follows from Eqs. (9) and (20):

$$\begin{aligned} \vec{A}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) &= \frac{1}{4\pi} (\vec{I} - \hat{\mathbf{n}}^{\text{sca}} \otimes \hat{\mathbf{n}}^{\text{sca}}) \cdot \int_{V_{\text{INT}}} d\mathbf{r}' \exp(-ik_1\hat{\mathbf{n}}^{\text{sca}} \cdot \mathbf{r}') \\ &\quad \times \int_{V_{\text{INT}}} d\mathbf{r}'' \vec{T}(\mathbf{r}', \mathbf{r}'') \exp(ik_1\hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{r}'') \cdot (\vec{I} - \hat{\mathbf{n}}^{\text{inc}} \otimes \hat{\mathbf{n}}^{\text{inc}}). \end{aligned} \quad (21)$$

The elements of the scattering dyadic have the dimension of length and are independent of the distance r to the observation point. However, \vec{A} depends on the exact choice of the origin O with respect to the scattering object. The dependence of the scattering dyadic on the absorption properties of the host medium enters through a non-zero value of k_1'' in Eqs. (21) and (10) and through the relative refractive index in Eq. (10).

Equation (20) shows that only four out of the nine components of the scattering dyadic are independent in the spherical polar coordinate system centered at the origin, Fig. 1. It is,

therefore, convenient to introduce the 2×2 amplitude scattering matrix \mathbf{S} describing the transformation of the θ - and φ -components of the electric field vector of the incident plane wave into those of the scattered spherical wave:

$$\mathbf{E}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) = \frac{\exp(ik_1 r)}{r} \mathbf{S}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \mathbf{E}_0^{\text{inc}}, \quad (22)$$

where \mathbf{E} denotes a two-element column formed by the θ - and φ -components of the electric field vector:

$$\mathbf{E} = \begin{bmatrix} E_\theta \\ E_\varphi \end{bmatrix}. \quad (23)$$

The elements of the amplitude scattering matrix have the dimension of length and are expressed in terms of the scattering dyadic as follows:

$$S_{11} = \hat{\boldsymbol{\theta}}^{\text{sca}} \cdot \vec{\tilde{A}} \cdot \hat{\boldsymbol{\theta}}^{\text{inc}}, \quad (24)$$

$$S_{12} = \hat{\boldsymbol{\theta}}^{\text{sca}} \cdot \vec{\tilde{A}} \cdot \hat{\boldsymbol{\phi}}^{\text{inc}}, \quad (25)$$

$$S_{21} = \hat{\boldsymbol{\phi}}^{\text{sca}} \cdot \vec{\tilde{A}} \cdot \hat{\boldsymbol{\theta}}^{\text{inc}}, \quad (26)$$

$$S_{22} = \hat{\boldsymbol{\phi}}^{\text{sca}} \cdot \vec{\tilde{A}} \cdot \hat{\boldsymbol{\phi}}^{\text{inc}}. \quad (27)$$

From now on, I will assume that the scattering dyadic and the scattering amplitude matrix can be computed for a given scattering object by solving the differential Maxwell equations (subject to appropriate boundary conditions) or their integral counterparts. Although many solution techniques discussed in [13, 19–22] have been traditionally applied to objects embedded in a non-absorbing host medium, the requisite extension to the case of an absorbing host medium is usually straightforward (cf. [3, 23, 24]).

4. Electromagnetic power

Let us now consider the classical scattering situation shown in Fig. 2. A finite fixed object is centered at the origin of the laboratory coordinate system and is illuminated by a plane electromagnetic wave incident in the direction of the unit vector $\hat{\mathbf{n}}^{\text{inc}}$. Polarization-sensitive detectors are located in the far-field zone of the object. The sensitive area of each detector is modeled as a plane circular surface element S normal to and centered at a position vector \mathbf{r} . Each detector is assumed to be well collimated, which means that electromagnetic energy incident on any point of the respective sensitive area S is detected only if the corresponding propagation direction falls within a narrow acceptance solid angle Ω centered around $\hat{\mathbf{r}}$. I also assume that the diameter of the sensitive area S is significantly greater than any linear dimension a of the scattering object: $D \gg a$. This will ensure that the right-hand side of Eq. (37) below is positive, thereby making possible a meaningful measurement of extinction (see also [25]). It is also assumed that the angular size of the sensitive area of a detector as viewed from the scattering object is small so that the scattered light impinging on different parts of S propagates in approximately the same direction. This is equivalent to requiring that $r \gg D/2$. Furthermore, it is assumed that the solid angle subtended by the sensitive area as viewed from the scattering object is smaller than the detector angular aperture: $S/r^2 < \Omega$. This ensures that all radiation scattered by the object in radial directions and impinging on S is detected. Detector 1 in Fig. 2 is centered at the incidence direction, whereas detector 2 is oriented such that the incidence direction does not fall within its acceptance solid angle: $\hat{\mathbf{n}}^{\text{inc}} \notin \Omega_2$.

The time-averaged Poynting vector $\langle \mathbf{S}(\mathbf{r}', t) \rangle_t$ at any point of the sensitive surface of a detector is the sum of three terms:

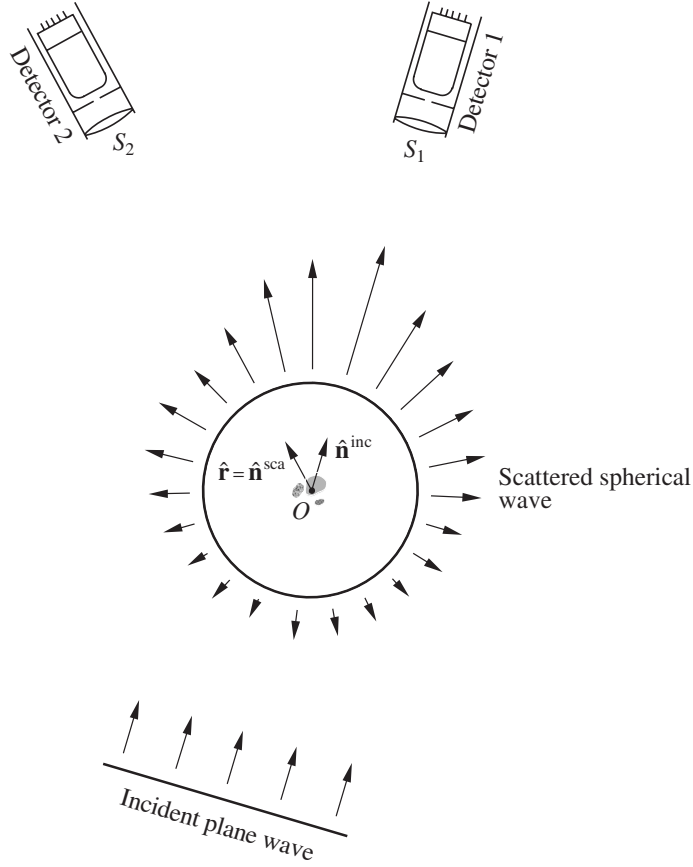


Fig. 2. Far-field optical observables.

$$\begin{aligned}\langle S(\mathbf{r}', t) \rangle_t &= \frac{1}{2} \text{Re} \{ \mathbf{E}(\mathbf{r}') \times [\mathbf{H}(\mathbf{r}')]^* \} \\ &= \langle S^{\text{inc}}(\mathbf{r}', t) \rangle_t + \langle S^{\text{sca}}(\mathbf{r}', t) \rangle_t + \langle S^{\text{ext}}(\mathbf{r}', t) \rangle_t,\end{aligned}\quad (28)$$

where $\mathbf{r}' = r' \hat{\mathbf{r}}'$ is the corresponding position vector,

$$\langle S^{\text{inc}}(\mathbf{r}', t) \rangle_t = \frac{1}{2} \text{Re} \{ \mathbf{E}^{\text{inc}}(\mathbf{r}') \times [\mathbf{H}^{\text{inc}}(\mathbf{r}')]^* \} \quad (29)$$

and

$$\langle S^{\text{sca}}(\mathbf{r}', t) \rangle_t = \frac{1}{2} \text{Re} \{ \mathbf{E}^{\text{sca}}(\mathbf{r}') \times [\mathbf{H}^{\text{sca}}(\mathbf{r}')]^* \} \quad (30)$$

are the Poynting vector components associated with the incident and the scattered field, respectively, and

$$\langle S^{\text{ext}}(\mathbf{r}', t) \rangle_t = \frac{1}{2} \text{Re} \{ \mathbf{E}^{\text{inc}}(\mathbf{r}') \times [\mathbf{H}^{\text{sca}}(\mathbf{r}')]^* + \mathbf{E}^{\text{sca}}(\mathbf{r}') \times [\mathbf{H}^{\text{inc}}(\mathbf{r}')]^* \} \quad (31)$$

can be interpreted as the term caused by interaction between the incident and scattered fields [8].

We have for the incident wave in the far-field zone of the scattering object (cf. Appendix B of [14]):

$$\begin{aligned}\mathbf{E}^{\text{inc}}(\mathbf{r}') &= \exp(ik_1 \hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{r}') \mathbf{E}_0^{\text{inc}} \\ &\stackrel{k_1 r' \rightarrow \infty}{=} \exp(-k_1'' \hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{r}') \frac{i2\pi}{k_1'} \left[\delta(\hat{\mathbf{n}}^{\text{inc}} + \hat{\mathbf{r}}') \frac{\exp(-ik_1' r')}{r'} - \delta(\hat{\mathbf{n}}^{\text{inc}} - \hat{\mathbf{r}}') \frac{\exp(ik_1' r')}{r'} \right] \mathbf{E}_0^{\text{inc}},\end{aligned}\quad (32)$$

$$\begin{aligned}\mathbf{H}^{\text{inc}}(\mathbf{r}') &= \frac{k_1}{\omega\mu_1} \exp(ik_1 \hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{r}') \hat{\mathbf{n}}^{\text{inc}} \times \mathbf{E}_0^{\text{inc}} \\ &\stackrel{k_1 r' \rightarrow \infty}{=} \exp(-k_1'' \hat{\mathbf{n}}^{\text{inc}} \cdot \mathbf{r}') \frac{i2\pi}{k_1'} \left[\delta(\hat{\mathbf{n}}^{\text{inc}} + \hat{\mathbf{r}}') \frac{\exp(-ik_1' r')}{r'} - \delta(\hat{\mathbf{n}}^{\text{inc}} - \hat{\mathbf{r}}') \frac{\exp(ik_1' r')}{r'} \right] \\ &\quad \times \frac{k_1}{\omega\mu_1} \hat{\mathbf{n}}^{\text{inc}} \times \mathbf{E}_0^{\text{inc}},\end{aligned}\quad (33)$$

while the transverse scattered spherical wave is given by

$$\mathbf{E}^{\text{sca}}(\mathbf{r}') = \frac{\exp(ik_1 r')}{r'} \mathbf{E}_1^{\text{sca}}(\hat{\mathbf{r}}'), \quad (34)$$

$$\mathbf{H}^{\text{sca}}(\mathbf{r}') = \frac{k_1}{\omega\mu_1} \frac{\exp(ik_1 r')}{r'} \hat{\mathbf{r}}' \times \mathbf{E}_1^{\text{sca}}(\hat{\mathbf{r}}'). \quad (35)$$

One can now derive that the total electromagnetic power received by detector 2 is given by

$$W_2(\hat{\mathbf{r}}) = \int_{S_2} dS \hat{\mathbf{r}} \cdot \langle \mathbf{S}(\mathbf{r}', t) \rangle_t \approx S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \frac{\exp(-2k_1'' r)}{r^2} |\mathbf{E}_1^{\text{sca}}(\hat{\mathbf{r}})|^2, \quad (36)$$

whereas for the exact forward-scattering direction (detector 1) we have

$$\begin{aligned}W_1(\hat{\mathbf{n}}^{\text{inc}}) &= \int_{S_1} dS \hat{\mathbf{n}}^{\text{inc}} \cdot \langle \mathbf{S}(\mathbf{r}', t) \rangle_t \\ &= S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \exp(-2k_1'' r) |\mathbf{E}_0^{\text{inc}}|^2 + \int_{S_1} dS \hat{\mathbf{n}}^{\text{inc}} \cdot [\langle \mathbf{S}^{\text{sca}}(\mathbf{r}', t) \rangle_t + \langle \mathbf{S}^{\text{ext}}(\mathbf{r}', t) \rangle_t] \\ &\approx S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \exp(-2k_1'' r) |\mathbf{E}_0^{\text{inc}}|^2 + S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \frac{\exp(-2k_1'' r)}{r^2} |\mathbf{E}_1^{\text{sca}}(\hat{\mathbf{n}}^{\text{inc}})|^2 \\ &\quad + r^2 \int_{\tilde{\Omega}_1} d\hat{\mathbf{r}}' \hat{\mathbf{n}}^{\text{inc}} \cdot \langle \mathbf{S}^{\text{ext}}(r\hat{\mathbf{r}}', t) \rangle_t \\ &\approx S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \exp(-2k_1'' r) |\mathbf{E}_0^{\text{inc}}|^2 + S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \frac{\exp(-2k_1'' r)}{r^2} |\mathbf{E}_1^{\text{sca}}(\hat{\mathbf{n}}^{\text{inc}})|^2 \\ &\quad - \exp(-2k_1'' r) \operatorname{Re} \left(\frac{k_1}{\omega\mu_1} \right) \frac{2\pi}{k_1'} \operatorname{Im}[\mathbf{E}_1^{\text{sca}}(\hat{\mathbf{n}}^{\text{inc}}) \cdot (\mathbf{E}_0^{\text{inc}})^*],\end{aligned}\quad (37)$$

where $\tilde{\Omega}_1$ is the solid angle centered around the direction $\hat{\mathbf{n}}^{\text{inc}}$ and subtended by the detector 1 surface at the distance r from the particle.

Equation (37) generalizes the optical theorem to the case of an absorbing host medium. The first term on the right-hand side is proportional to the detector area and is equal to the electromagnetic power that would be received by detector 1 in the absence of the scattering object. The third term is independent of S and describes attenuation caused by interposing the object between the light source and the detector. Thus, the detector centered at the exact

forward-scattering direction measures the power of the incident light attenuated by the interference of the incident and scattered fields as well as a relatively small contribution from the scattered light. The detector centered at any other direction registers only the scattered light.

5. Backscattering interference

The presence of the terms proportional to $\delta(\hat{\mathbf{n}}^{\text{inc}} + \hat{\mathbf{r}}')$ in Eqs. (32) and (33) makes it instructive to also consider the flow of electromagnetic energy through a surface element S_3 normal to the backscattering direction and centered at the position vector $-r\hat{\mathbf{n}}^{\text{inc}}$. The orientation of the surface element is given by the unit vector $-\hat{\mathbf{n}}^{\text{inc}}$. It is straightforward to derive that the corresponding total electromagnetic power is given by

$$\begin{aligned} W_3(-\hat{\mathbf{n}}^{\text{inc}}) &= - \int_{S_3} dS \hat{\mathbf{n}}^{\text{inc}} \cdot \langle \mathbf{S}(\mathbf{r}', t) \rangle_t \\ &\approx -S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \exp(2k_1''r) |\mathbf{E}_0^{\text{inc}}|^2 + S \operatorname{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \frac{\exp(-2k_1''r)}{r^2} |\mathbf{E}_1^{\text{sca}}(-\hat{\mathbf{n}}^{\text{inc}})|^2 \\ &\quad + \operatorname{Im} \left(\frac{k_1}{\omega\mu_1} \right) \frac{2\pi}{k_1'} \operatorname{Re} [\exp(i2k_1'r) \mathbf{E}_1^{\text{sca}}(-\hat{\mathbf{n}}^{\text{inc}}) \cdot (\mathbf{E}_0^{\text{inc}})^*]. \end{aligned} \quad (38)$$

This formula is quite interesting. The first term describes the flow of the incident energy through S_3 and is negative since the direction of the energy flow is opposite to the orientation of the surface element. The absolute value of this term increases exponentially with distance from the scattering object. The second term is the power backscattered by the object. The third term has no monotonous independence on r and describes the interference of the incident and backscattered waves [26]; it vanishes if the host medium is non-absorbing [27]. The result of the interference is a standing wave [28] which, because of the rapidly oscillating factor $\exp(i2k_1'r)$, does not cause a long-range transport of electromagnetic energy. However, there is a stationary local transport of energy within wavelength-long elementary volume elements centered at the straight line extending from the origin of the coordinate system in the direction $-\hat{\mathbf{n}}^{\text{inc}}$. Since the solution of the frequency-domain scattering problem is time-independent, there may not be infinite accumulation of energy at any point in the medium. Therefore, energy transported from one point of a wavelength-long volume element to another must be absorbed by the host medium. This means that the backscattering interference causes additional absorption of electromagnetic energy at points along the straight line extending in the exact backscattering direction.

Although the backscattering interference is, no doubt, a real physical effect, its practical importance is limited. Indeed, while the first and third terms on the right-hand-side of Eq. (37) are attenuated equally by absorption in the host medium, there is a notable difference between the first and third terms on the right-hand side of Eq. (38). In a moderately or strongly absorbing host medium, the first term dominates the local flow of energy because of the rapidly growing exponential factor $\exp(2k_1''r)$. In a weakly absorbing medium, the third term can usually be neglected because $\operatorname{Im}(k_1/\mu_1)$ is much smaller than $\operatorname{Re}(k_1/\mu_1)$.

6. Phase matrix

Since both the incident homogeneous plane wave and the scattered outgoing spherical wave are transverse, we can introduce the corresponding coherency column vectors, \mathbf{J} , and Stokes column vectors, \mathbf{I} , as follows:

$$\mathbf{J}^{\text{inc}} = \text{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \begin{bmatrix} E_{0\theta}^{\text{inc}} (E_{0\theta}^{\text{inc}})^* \\ E_{0\theta}^{\text{inc}} (E_{0\varphi}^{\text{inc}})^* \\ E_{0\varphi}^{\text{inc}} (E_{0\theta}^{\text{inc}})^* \\ E_{0\varphi}^{\text{inc}} (E_{0\varphi}^{\text{inc}})^* \end{bmatrix}, \quad (39)$$

$$\begin{aligned} \mathbf{J}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) &= \text{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \begin{bmatrix} E_{\theta}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) [E_{\theta}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}})]^* \\ E_{\theta}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) [E_{\varphi}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}})]^* \\ E_{\varphi}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) [E_{\theta}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}})]^* \\ E_{\varphi}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) [E_{\varphi}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}})]^* \end{bmatrix} \\ &= \frac{\exp(-2k_1''r)}{r^2} \text{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \begin{bmatrix} E_{1\theta}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}}) [E_{1\theta}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}})]^* \\ E_{1\theta}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}}) [E_{1\varphi}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}})]^* \\ E_{1\varphi}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}}) [E_{1\theta}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}})]^* \\ E_{1\varphi}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}}) [E_{1\varphi}^{\text{sca}}(\hat{\mathbf{n}}^{\text{sca}})]^* \end{bmatrix}, \quad (40) \end{aligned}$$

$$\mathbf{I}^{\text{inc}} = \mathbf{D} \mathbf{J}^{\text{inc}} = \text{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \begin{bmatrix} E_{0\theta}^{\text{inc}} (E_{0\theta}^{\text{inc}})^* + E_{0\varphi}^{\text{inc}} (E_{0\varphi}^{\text{inc}})^* \\ E_{0\theta}^{\text{inc}} (E_{0\theta}^{\text{inc}})^* - E_{0\varphi}^{\text{inc}} (E_{0\varphi}^{\text{inc}})^* \\ -E_{0\theta}^{\text{inc}} (E_{0\varphi}^{\text{inc}})^* - E_{0\varphi}^{\text{inc}} (E_{0\theta}^{\text{inc}})^* \\ i[E_{0\varphi}^{\text{inc}} (E_{0\theta}^{\text{inc}})^* - E_{0\theta}^{\text{inc}} (E_{0\varphi}^{\text{inc}})^*] \end{bmatrix}, \quad (41)$$

$$\mathbf{I}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) = \mathbf{D} \mathbf{J}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) = \frac{\exp(-2k_1''r)}{r^2} \text{Re} \left(\frac{k_1}{2\omega\mu_1} \right) \begin{bmatrix} E_{1\theta}^{\text{sca}} (E_{1\theta}^{\text{sca}})^* + E_{1\varphi}^{\text{sca}} (E_{1\varphi}^{\text{sca}})^* \\ E_{1\theta}^{\text{sca}} (E_{1\theta}^{\text{sca}})^* - E_{1\varphi}^{\text{sca}} (E_{1\varphi}^{\text{sca}})^* \\ -E_{1\theta}^{\text{sca}} (E_{1\varphi}^{\text{sca}})^* - E_{1\varphi}^{\text{sca}} (E_{1\theta}^{\text{sca}})^* \\ i[E_{1\varphi}^{\text{sca}} (E_{1\theta}^{\text{sca}})^* - E_{1\theta}^{\text{sca}} (E_{1\varphi}^{\text{sca}})^*] \end{bmatrix}, \quad (42)$$

where

$$\mathbf{D} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ 0 & -1 & -1 & 0 \\ 0 & -i & i & 0 \end{bmatrix}. \quad (43)$$

According to their definition, the elements of the Stokes column vectors are always real-valued. The first Stokes parameter, I , gives the total intensity of a wave. It is straightforward to show that

$$\mathbf{J}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) = \frac{\exp(-2k_1''r)}{r^2} \mathbf{Z}^J(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \mathbf{J}^{\text{inc}} \quad (44)$$

and

$$\mathbf{I}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) = \frac{\exp(-2k_1''r)}{r^2} \mathbf{Z}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \mathbf{I}^{\text{inc}}, \quad (45)$$

where the elements of the corresponding phase matrices are expressed in terms of the amplitude matrix elements $S_{ij}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}})$ as follows:

$$\mathbf{Z}^J = \begin{bmatrix} |S_{11}|^2 & S_{11}S_{12}^* & S_{12}S_{11}^* & |S_{12}|^2 \\ S_{11}S_{21}^* & S_{11}S_{22}^* & S_{12}S_{21}^* & S_{12}S_{22}^* \\ S_{21}S_{11}^* & S_{21}S_{12}^* & S_{22}S_{11}^* & S_{22}S_{12}^* \\ |S_{21}|^2 & S_{21}S_{22}^* & S_{22}S_{21}^* & |S_{22}|^2 \end{bmatrix}, \quad (46)$$

$$\mathbf{Z}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) = \mathbf{D}\mathbf{Z}^J(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}})\mathbf{D}^{-1} \quad (47)$$

or, explicitly,

$$Z_{11} = \frac{1}{2}(|S_{11}|^2 + |S_{12}|^2 + |S_{21}|^2 + |S_{22}|^2), \quad (48)$$

$$Z_{12} = \frac{1}{2}(|S_{11}|^2 - |S_{12}|^2 + |S_{21}|^2 - |S_{22}|^2), \quad (49)$$

$$Z_{13} = -\text{Re}(S_{11}S_{12}^* + S_{22}S_{21}^*), \quad (50)$$

$$Z_{14} = -\text{Im}(S_{11}S_{12}^* - S_{22}S_{21}^*), \quad (51)$$

$$Z_{21} = \frac{1}{2}(|S_{11}|^2 + |S_{12}|^2 - |S_{21}|^2 - |S_{22}|^2), \quad (52)$$

$$Z_{22} = \frac{1}{2}(|S_{11}|^2 - |S_{12}|^2 - |S_{21}|^2 + |S_{22}|^2), \quad (53)$$

$$Z_{23} = -\text{Re}(S_{11}S_{12}^* - S_{22}S_{21}^*), \quad (54)$$

$$Z_{24} = -\text{Im}(S_{11}S_{12}^* + S_{22}S_{21}^*), \quad (55)$$

$$Z_{31} = -\text{Re}(S_{11}S_{21}^* + S_{22}S_{12}^*), \quad (56)$$

$$Z_{32} = -\text{Re}(S_{11}S_{21}^* - S_{22}S_{12}^*), \quad (57)$$

$$Z_{33} = \text{Re}(S_{11}S_{22}^* + S_{12}S_{21}^*), \quad (58)$$

$$Z_{34} = \text{Im}(S_{11}S_{22}^* + S_{12}S_{21}^*), \quad (59)$$

$$Z_{41} = -\text{Im}(S_{21}S_{11}^* + S_{22}S_{12}^*), \quad (60)$$

$$Z_{42} = -\text{Im}(S_{21}S_{11}^* - S_{22}S_{12}^*), \quad (61)$$

$$Z_{43} = \text{Im}(S_{22}S_{11}^* - S_{12}S_{21}^*), \quad (62)$$

$$Z_{44} = \text{Re}(S_{22}S_{11}^* - S_{12}S_{21}^*). \quad (63)$$

The elements of both phase matrices have the dimension of length and are independent of the distance from the origin of the laboratory coordinate system to the observation point. The elements of the Stokes phase matrix are real-valued.

Let us now assume that both detectors of electromagnetic radiation in Fig. 2 are polarization sensitive and allow one to measure all four Stokes parameters of light impinging on them. By analogy with Eq. (36), the polarized reading of detector 2 is given by

$$\text{Signal 2} \approx \Delta S \mathbf{I}^{\text{sca}}(r\hat{\mathbf{n}}^{\text{sca}}) = \frac{\exp(-2k_1''r)}{r^2} \mathbf{Z}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \mathbf{I}^{\text{inc}}. \quad (64)$$

Note that \mathbf{I}^{inc} is the Stokes column vector of the incident wave at the origin of the laboratory coordinate system.

7. Extinction matrix

Let us now consider the special case of the exact forward-scattering direction ($\hat{\mathbf{r}} = \hat{\mathbf{n}}^{\text{inc}}$). Because now both the incident homogeneous plane wave and the scattered outgoing spherical wave propagate in the same direction and are transverse, their superposition is also a

transverse wave propagating in the forward direction. Therefore, we can define the coherency column vector of the total field for propagation directions $\hat{\mathbf{r}}$ very close to $\hat{\mathbf{n}}^{\text{inc}}$ as follows:

$$\mathbf{J}(r\hat{\mathbf{r}}) = \text{Re} \left(\frac{k_1}{2\omega\mu_1} \begin{bmatrix} E_\theta(r\hat{\mathbf{r}})[E_\theta(r\hat{\mathbf{r}})]^* \\ E_\theta(r\hat{\mathbf{r}})[E_\phi(r\hat{\mathbf{r}})]^* \\ E_\phi(r\hat{\mathbf{r}})[E_\theta(r\hat{\mathbf{r}})]^* \\ E_\phi(r\hat{\mathbf{r}})[E_\phi(r\hat{\mathbf{r}})]^* \end{bmatrix} \right), \quad (65)$$

where the total electric field is given by

$$\mathbf{E}(r\hat{\mathbf{r}}) = \mathbf{E}^{\text{inc}}(r\hat{\mathbf{r}}) + \mathbf{E}^{\text{sca}}(r\hat{\mathbf{r}}). \quad (66)$$

Integrating the elements of $\mathbf{J}(r\hat{\mathbf{r}})$ over the surface of the collimated detector aligned normal to $\hat{\mathbf{n}}^{\text{inc}}$ and using Eqs. (32) and (34), we derive for the coherency-vector representation of the polarized signal recorded by detector 1 in Fig. 2:

$$\begin{aligned} (\text{Signal } 1)^J &= \int_{S_1} dS \mathbf{J}(r\hat{\mathbf{r}}) \\ &\approx S \exp(-2k_1''r) \mathbf{J}^{\text{inc}} + S \frac{\exp(-2k_1''r)}{r^2} \mathbf{Z}^J(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}}) \mathbf{J}^{\text{inc}} - \exp(-2k_1''r) \mathbf{K}^J(\hat{\mathbf{n}}^{\text{inc}}) \mathbf{J}^{\text{inc}}, \end{aligned} \quad (67)$$

where $\mathbf{Z}^J(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}})$ is the forward-scattering coherency phase matrix, and the elements of the 4×4 coherency extinction matrix $\mathbf{K}^J(\hat{\mathbf{n}}^{\text{inc}})$ are expressed in terms of the elements of the forward-scattering amplitude matrix $\mathbf{S}(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}})$ as follows:

$$\mathbf{K}^J = \frac{i2\pi}{k_1'} \begin{bmatrix} S_{11}^* - S_{11} & S_{12}^* & -S_{12} & 0 \\ S_{21}^* & S_{22}^* - S_{11} & 0 & -S_{12} \\ -S_{21} & 0 & S_{11}^* - S_{22} & S_{12}^* \\ 0 & -S_{21} & S_{21}^* & S_{22}^* - S_{22} \end{bmatrix}. \quad (68)$$

In the Stokes-vector representation,

$$\begin{aligned} \text{Signal } 1 &= \int_{S_1} dS \mathbf{l}(r\hat{\mathbf{r}}) \\ &\approx S \exp(-2k_1''r) \mathbf{l}^{\text{inc}} + S \frac{\exp(-2k_1''r)}{r^2} \mathbf{Z}(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}}) \mathbf{l}^{\text{inc}} - \exp(-2k_1''r) \mathbf{K}(\hat{\mathbf{n}}^{\text{inc}}) \mathbf{l}^{\text{inc}}, \end{aligned} \quad (69)$$

where

$$\mathbf{l}(r\hat{\mathbf{n}}^{\text{inc}}) = \mathbf{D} \mathbf{J}(r\hat{\mathbf{n}}^{\text{inc}}). \quad (70)$$

The 4×4 Stokes extinction matrix $\mathbf{K}(\hat{\mathbf{n}}^{\text{inc}})$ is given by

$$\mathbf{K}(\hat{\mathbf{n}}^{\text{inc}}) = \mathbf{D} \mathbf{K}^J(\hat{\mathbf{n}}^{\text{inc}}) \mathbf{D}^{-1}. \quad (71)$$

The explicit formulas for the elements of this matrix in terms of the elements of the forward-scattering amplitude matrix $\mathbf{S}(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}})$ are as follows:

$$K_{jj} = \frac{2\pi}{k_1'} \text{Im}(S_{11} + S_{22}), \quad j = 1, \dots, 4, \quad (72)$$

$$K_{12} = K_{21} = \frac{2\pi}{k_1'} \text{Im}(S_{11} - S_{22}), \quad (73)$$

$$K_{13} = K_{31} = -\frac{2\pi}{k'_1} \text{Im}(S_{12} + S_{21}), \quad (74)$$

$$K_{14} = K_{41} = \frac{2\pi}{k'_1} \text{Re}(S_{21} - S_{12}), \quad (75)$$

$$K_{23} = -K_{32} = \frac{2\pi}{k'_1} \text{Im}(S_{21} - S_{12}), \quad (76)$$

$$K_{24} = -K_{42} = -\frac{2\pi}{k'_1} \text{Re}(S_{12} + S_{21}), \quad (77)$$

$$K_{34} = -K_{43} = \frac{2\pi}{k'_1} \text{Re}(S_{22} - S_{11}). \quad (78)$$

The elements of the coherency and Stokes extinction matrices have the dimension of area and are independent of the distance r to the observation point.

8. Reciprocity and backscattering symmetry

The reader can easily verify that Eq. (2.55) of [13] and Eq. (3.4.13) of [14] are quite general and are valid for absorbing as well as non-absorbing host media. Furthermore, the only change in Eq. (2.61) of [13] and Eq. (3.4.18) of [14] is to replace k_1 with k'_1 . Therefore, the resulting reciprocity relations for the scattering dyadic [29], the amplitude scattering matrix, and the phase and extinction matrices remain the same as in the case of a non-absorbing host medium:

$$\vec{\vec{A}}(-\hat{\mathbf{n}}^{\text{inc}}, -\hat{\mathbf{n}}^{\text{sca}}) = [\vec{\vec{A}}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}})]^T, \quad (79)$$

$$\mathbf{S}(-\hat{\mathbf{n}}^{\text{inc}}, -\hat{\mathbf{n}}^{\text{sca}}) = \begin{bmatrix} S_{11}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) & -S_{21}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \\ -S_{12}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) & S_{22}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}}) \end{bmatrix}, \quad (80)$$

$$\mathbf{Z}(-\hat{\mathbf{n}}^{\text{inc}}, -\hat{\mathbf{n}}^{\text{sca}}) = \mathbf{\Delta}_3 [\mathbf{Z}(\hat{\mathbf{n}}^{\text{sca}}, \hat{\mathbf{n}}^{\text{inc}})]^T \mathbf{\Delta}_3, \quad (81)$$

$$\mathbf{K}(-\hat{\mathbf{n}}^{\text{inc}}) = \mathbf{\Delta}_3 [\mathbf{K}(\hat{\mathbf{n}}^{\text{inc}})]^T \mathbf{\Delta}_3, \quad (82)$$

where

$$\mathbf{\Delta}_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (83)$$

and T denotes a transposed matrix. The related backscattering symmetry properties

$$S_{21}(-\hat{\mathbf{n}}, \hat{\mathbf{n}}) = -S_{12}(-\hat{\mathbf{n}}, \hat{\mathbf{n}}), \quad (84)$$

$$Z_{11}(-\hat{\mathbf{n}}, \hat{\mathbf{n}}) - Z_{22}(-\hat{\mathbf{n}}, \hat{\mathbf{n}}) + Z_{33}(-\hat{\mathbf{n}}, \hat{\mathbf{n}}) - Z_{44}(-\hat{\mathbf{n}}, \hat{\mathbf{n}}) = 0, \quad (85)$$

and

$$\mathbf{K}(-\hat{\mathbf{n}}^{\text{inc}}) = \begin{bmatrix} K_{11}(\hat{\mathbf{n}}^{\text{inc}}) & K_{12}(\hat{\mathbf{n}}^{\text{inc}}) & -K_{13}(\hat{\mathbf{n}}^{\text{inc}}) & K_{14}(\hat{\mathbf{n}}^{\text{inc}}) \\ K_{21}(\hat{\mathbf{n}}^{\text{inc}}) & K_{22}(\hat{\mathbf{n}}^{\text{inc}}) & K_{23}(\hat{\mathbf{n}}^{\text{inc}}) & -K_{24}(\hat{\mathbf{n}}^{\text{inc}}) \\ -K_{31}(\hat{\mathbf{n}}^{\text{inc}}) & K_{32}(\hat{\mathbf{n}}^{\text{inc}}) & K_{33}(\hat{\mathbf{n}}^{\text{inc}}) & K_{34}(\hat{\mathbf{n}}^{\text{inc}}) \\ K_{41}(\hat{\mathbf{n}}^{\text{inc}}) & -K_{42}(\hat{\mathbf{n}}^{\text{inc}}) & K_{43}(\hat{\mathbf{n}}^{\text{inc}}) & K_{44}(\hat{\mathbf{n}}^{\text{inc}}) \end{bmatrix} \quad (86)$$

remain unchanged as well.

9. Discussion

The formulas derived in this paper generalize those appearing in the theory of single scattering by a fixed object embedded in a non-absorbing medium [8, 13, 14]. In fact, the former can be reduced to the latter by setting $\text{Im}\varepsilon_1 = \text{Im}\mu_1 = k_1'' = 0$ and $k_1' = k_1$.

The generalized VIE (8) as well as the generalized equation for the dyadic transition operator, Eq. (10), fully retain their formal mathematical structure. However, their solutions are now affected explicitly by non-zero absorptivity of the host medium.

The dyadic transition operator is independent of the amplitude and propagation direction of the incident field, while still being dependent on the complex wave number of the host medium. This means that \vec{T} is independent of the specific location and orientation of the scattering object with respect to the incident field. It depends, however, on the position and orientation of the object with respect to the laboratory coordinate system.

An absorbing host medium can support inhomogeneous plane waves for which the real and imaginary components of the wave vector are not parallel. Since the definition of the Stokes parameters for such waves is problematic, our discussion of far-field scattering is limited to the case of illumination by a homogeneous plane wave. The generalization to the case of illumination by a field expandable in homogeneous plane waves is quite straightforward.

Equations (11), (12), and (15) generalize the conditions of applicability of the far-field approximation derived in [30] for the case of a non-absorbing host medium. Although the corresponding modifications may appear to be insignificant mathematically, the practical implications of Eqs. (11), (12), and (15) can be quite dramatic if the host medium is even moderately absorbing. Indeed, then the requisite minimal distance between the object and the observation point may turn out to be large enough to essentially extinguish any observable manifestation of electromagnetic scattering. Thus Eqs. (11), (12), and (15) can often be expected to render the very concept of far-field scattering practically useless, unless k_1'' remains sufficiently small.

The scattering dyadic and the scattering amplitude matrix are naturally defined in such a way that they are independent of the amplitude of the incident field and, thus, of the position of the object with respect to the incident field. They are also independent of the distance to the observation point, but depend on the incidence and scattering directions. This way of defining the scattering dyadic also serves to preserve the fundamental Saxon's reciprocity relation.

The transversality of the incident plane wave and the scattered spherical wave allows for a straightforward generalization of the definition of the Stokes parameters as direct optical observables. The quantities directly describing the response of far-field polarization-sensitive detectors of electromagnetic energy flow are the phase and extinction matrices. They are expressed in terms of the amplitude scattering matrix and as such are independent of the amplitude of the incident field and of the distance to the observation point, but depend on the incidence and scattering directions. They also satisfy the conventional reciprocity relations.

It follows from Eqs. (64) and (69) that non-zero absorptivity of the host medium affects the result of a far-field optical measurement in two ways. First, it attenuates both the incident light and the scattered light on their way to the detector. Second, it modifies the values of the phase and extinction matrix elements. One can expect that in many cases the first effect is likely to be more important. Indeed, if absorption is strong enough to affect the optical properties of a wavelength-sized particle then it should be expected to be strong enough to extinguish any practically observable manifestation of scattering at a multi-wavelength distance from the particle to the detector.

For a spherically symmetric particle, $S_{11}(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}}) = S_{22}(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}}) \equiv S_{11}(0^\circ)$ and $S_{12}(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}}) = S_{21}(\hat{\mathbf{n}}^{\text{inc}}, \hat{\mathbf{n}}^{\text{inc}}) \equiv 0$. Therefore, the third term on the right-hand side of Eq. (69)

reduces to $-\exp(-2k_1''r)C_{\text{ext}}\mathbf{l}^{\text{inc}}$, where the spherical-particle extinction cross section is given by

$$C_{\text{ext}} = \frac{4\pi}{k_1'} \text{Im} S_{11}(0^\circ). \quad (87)$$

This formula corrects Eq. (11) of [23] and Eq. (8) of [4]. Indeed, taking into account that the definitions of the amplitude scattering matrix in this paper and in [23] differ by a constant factor i/k_1 , Eq. (11) of [23] can be re-written as $C_{\text{ext}} = 4\pi \text{Im}[S_{11}(0^\circ)/k_1]$, which is different from Eq. (87) above. The origin of the difference can be traced back to Eqs. (6) and (7) of [23] which are applicable only if the wave number in the host medium is real-valued. Since the latter is assumed to be complex in general, Eqs. (6) and (7) of [23] can be used only after separating out the real part of the wave number and making the exponent in Eq. (6) purely imaginary. This, obviously, leads to Eq. (87) above.

As a final remark we note that the assumption that the host absorbing medium is infinite has simplified all derivations considerably, but is a mathematical idealization in that the growth of the incident energy flux as one moves in the direction $-\hat{\mathbf{n}}^{\text{inc}}$ appears to be unlimited. Giving the medium a boundary located far from the scattering object and assuming that the source of the incident wave is located outside of this boundary offers a simple practical way out of this seemingly unphysical situation: the energy flow would increase exponentially with distance from the object until the medium's boundary is reached where it would then become constant.

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